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not use the modern exponents in any of his editions of his *Clavis mathematica* (London, 1631, 1648, 1652), but if Rigaud reproduced faithfully the notations in the original letters, it follows that Oughtred used positive integral exponents in his correspondence as early as 1642.¹ On Feb. 5, 1666–7, John Wallis wrote to John Collins, a proposed new edition of Oughtred's *Clavis* being under discussion: "It is true, that as in other things so in mathematics, fashions will daily alter, and that which Mr. Oughtred designed by great letters may now by others be designed by small; but a mathematician will, with the same ease and advantage, understand Ac and a^3 or aaa ."² John Pell wrote r^2 and t^2 in a letter written in Amsterdam on Aug. 7, 1645.³ Pascal made free use of positive integral exponents in several of his papers, particularly the *Potestatum numericarum summa*, 1654. G. Kinckhuysen used positive integral exponents in 1660, in his *Meet-Konst*, and in 1661 in his *Algebra*.⁴

[The February issue will conclude the discussion of the modern exponential notation and take up the unsuccessful attempts to create a theory of logarithms of negative numbers.]

ERRORS IN THE LITERATURE ON GROUPS OF FINITE ORDER.

By G. A. MILLER, University of Illinois.

ERRORS IN GENERAL.

Although mathematics is an exact science, the mathematical literature is disfigured by numerous errors. Some of the most influential works fail along this line. In the preface to volume 1 of his *Zahlentheorie*, 1892, Paul Bachmann observes that Gauss' *Disquisitiones arithmeticae*, 1801, was scarcely readable on account of the large number of annoying typographical errors.

Imperfect Proofs.—While typographical errors are annoying they are not the most serious errors. A much more annoying class of errors is due to imperfect proofs. The careless use of infinite series during the seventeenth and eighteenth centuries, without any inquiry as to convergence, is one of the most important instances of imperfect methods. In fact, in the early part of the nineteenth century there were only a few mathematicians who were sufficiently careful about this matter; so that Abel could properly write, in 1826, to his friend Holmboe, that most of the papers dealing with series were inexact.⁵ Recently Study called atten-

¹ S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century*, Vol. I, Oxford, 1841, p. 63.

² Rigaud, *op. cit.*, Vol. I, p. 475.

³ J. O. Halliwell, *Progress of Science in England*, London, 1841, p. 89.

⁴ Kinckhuysen, *De Grondt der Meet-Konst*, De Haerlem, 1660; *Algebra ofte Stel-Konst*, De Haerlem, 1661.

⁵ Niels Henrik Abel Memorial, Correspondance, 1902, p. 16.

tion to the great lack of precision in much of our recent geometrical production, referring to it as "indegesta moles geometrica."¹

A third large class of errors is due to the oversight of an important step in an otherwise perfect proof. Many of the supposed proofs of the parallel axiom, starting with those of Ptolomy and Proclus, are of this nature. This is also true of many of the supposed proofs of the fundamental theorem of algebra, including those associated with such prominent names as d'Alembert, Euler, Lagrange, and Laplace.² Many recent examples of such errors are found in the literature relating to attempts to prove the great Fermat theorem.

Historical References.—Another large class of errors relates to historical notes. As a result of the recent very rapid progress in the development of our knowledge of the history of mathematics a very large number of the older mathematical references have been proved to be incorrect. Even some of the most recent and most reliable books disclose a lack of sufficient care in this respect. As instances of this kind, we may mention the erroneous statements regarding the early use of the term function, given both in the *Fundamental Concepts of Algebra and Geometry* by J. W. Young, 1911, page 194, and also in the *Mono-graphs on Modern Mathematics* by J. W. A. Young, 1911, page 265.³ It is, of course, generally unnecessary to give historical references in a mathematical work, but when they are given they should be correct.

Lists of Errors.—In 1904 Maillet proposed, as question number 2855 in *L'Intermédiaire des Mathématiciens*, the making of a list of errors committed by prominent deceased mathematicians. Maillet began this list by referring to errors committed by Ampère, Abel, Chasles, Legendre, Bertrand, Laplace, and others. Many others have contributed to this list but it is still very incomplete. It is, however, sufficiently extensive to exhibit many interesting questions with important caution signals, and it is to be hoped that this list will be gradually extended. A similar list relating to the errors committed by the great living mathematicians would probably be still more useful.

The question of errors in mathematical literature is especially important at the present time in view of their rapid increase on account of the numerous efforts to prove Fermat's theorem, and in view of the great encyclopedias of mathematics which are in course of publication. Both of these encyclopedias invite the mathematical world to coöperate in the effort to make advances and to banish falsehoods. For this purpose the German encyclopedia established a "Sprechsaal" in the *Archiv der Mathematik und Physik* which was transferred in 1907 to the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, while the French edition publishes a "Tribune publique" in connection with the regular numbers of the encyclopedia. It is expected to embody all the valuable features of the "Tribune publique" in the "Additions et Modifications" which are to appear near the end of each volume. In view of the extensive and permanent circulation

¹ Study, *Archiv der Mathematik und Physik*, volume 18 (1911), p. 175.

² Cf. *Encyclopédie des Sciences Mathématiques*, tome 1, vol. 2, p. 193.

³ This error seems to have been started by d'Alembert. It has been corrected in the *Encyclopédie des Sciences Mathématiques*, tome II, vol. 1, p. 3.

of these encyclopedias it is extremely important that they should be as reliable as possible, and it is to be hoped that their appeal for public coöperation will lead to more and more valuable responses. It is almost as important that matters of no value be excluded from such standard works as it is that all matters which exhibit real progress should be included.

First Aid to Truth.—For the first correction of historical errors the *Bibliotheca Mathematica* affords an excellent medium. This journal has already rendered most valuable services by directing attention to the large number of errors in some of the most popular works on the history of mathematics, and by exposing these errors in a fearless manner. It should be remembered that errors are weeds and that they tend to impede the growth of the truth. It is one of the main duties of the true scholar to assist in their destruction, since this is essential for a perfect harvest. The man who fearlessly and unselfishly exposes errors, especially when they appear in the best books, is thereby rendering a great service, which is frequently not sufficiently appreciated.

Reviews constitute the most important single instrument to combat errors. Unfortunately reviewers do not always use this instrument wisely. In many cases this may be due to a lack of self-confidence on the part of the reviewer. In other cases it is doubtless due to the inclination to write a review which may please the author of the work under review. A conscientious reviewer cannot escape the sense of duty to the public and to truth. If we could have more reviews like those which Study, of Bonn, writes, we should doubtless have a more enlightened public and a larger number of careful authors.

Students of mathematics should read more semi-mathematical literature, such as these reviews, and the historical or philosophical mathematical literature. Many new viewpoints come to us in this way. Errors due to eminent mathematicians furnish an especially valuable topic of a semi-mathematical nature, and even if a student reads reviews of books and of articles mainly from a sense of curiosity as to how the work of those whom he knows is regarded by competent critics, it is very much better than if he reads nothing outside of those things which bear directly on his regular work. It should be remembered, however, that it is very easy to discover little errors in a great man's work, just as it is easy to make slight extensions to such work. The correction of really big and important errors is the work of a big man. A man discloses his caliber by the way he corrects errors just as much as by his other publications.

ERRORS IN THE EARLY LITERATURE OF GROUP THEORY.

Galois was the first to enter deeply into the theory of groups. In his efforts to develop a general theory of equations he directed attention to the important concept of simple groups and proved that the icosahedral group is the smallest simple group of composite order. He also studied a very important infinite system of simple groups, known as the general modular groups, of order $\frac{1}{2}p(p^2 - 1)$ and of degree $p + 1$, $p > 3$. In one of his earliest papers, published in the

Bulletin des Sciences Mathématiques de M. Férussac, 1830, page 271, he asserts that this modular group can be represented as a substitution group of degree p only when $p = 5$. Later he corrected his mistake by observing that this representation is also possible when $p = 7$ or 11, but for no larger value of p . A recent proof of this important theorem is due to Frobenius, who also observes that the modular group is the only simple group of order $\frac{1}{2}p(p^2 - 1)$,¹ $p > 3$.

Errors of Cauchy.—Cauchy was the first who wrote extensively on the theory of substitution groups, and he is commonly called the founder of this theory in view of his many valuable contributions along this line. Unfortunately, the number of errors in his writings is very large, and most of them are repeated in the published parts of his complete works. As these works are brought out under the auspices of the Paris Academy of Sciences, one might have expected footnotes calling attention to the errors which are so distracting and which impair so seriously the usefulness of the work; especially since the publication is proceeding very slowly, having commenced about thirty years ago. Not only do these writings involve many erroneous statements but they also contain obscurities which later developments have completely cleared away. Those who edit collected works do not always take sufficient care to avoid perpetuating the errors and obscurities of a writer.

Cauchy began the construction of fairly long lists of substitution groups which are supposed to contain all the possible substitution groups on six or a smaller number of letters. Such lists were based by Cauchy upon an insufficient knowledge of the theory of substitutions and hence they were incomplete. Very many later writers had a similar experience and the efforts to enumerate all the possible substitutions groups of certain degrees have led to a very large number of errors in the literature of group theory, especially on the part of Kirkman, Askwith, and Cayley. One of the most important instances of false theorems announced by Cauchy is the one which states that every simply transitive group of degree $p + 1$, p being a prime number, is imprimitive.²

Errors of Cayley.—Cayley's writings on group theory are scarcely more accurate than those of Cauchy. In volume 1 of the *American Journal of Mathematics*, page 51, Cayley states that there are three groups of order 6, giving as examples, the non-cyclic group and the cyclic group in two different forms. It seems very strange that Cayley should have made a mistake regarding such a simple matter more than twenty years after he began to study group theory. About thirteen years later Cayley attempted to enumerate the substitution groups³ whose degrees do not exceed 8 and in this work the number of errors is very large, as has been observed by various writers. These errors have been repeated in the *Collected Mathematical Papers*, volume XIII, 1897, and the very evident error in reference to the groups of order 6 is also repeated in volume X, 1896.

Errors of Jordan.—Jordan did very much pioneer work in group theory and he attacked some of the most difficult problems. While the number of errors in

¹ Frobenius, *Berliner Sitzungsberichte*, 1902, page 351.

² Cauchy, Paris, *Comptes Rendus*, volume 21 (1845), page 1199.

³ These enumerations appeared in *Quarterly Journal of Mathematics*, volume 25, 1891.

his works is considerable it is perhaps not larger than one could reasonably expect from the difficulty and newness of the problems. In particular, as early as 1878 he made the first attempt to determine all the finite collineation groups in three variables, but he failed to find the two important groups of orders 168 and 360, which were discovered later by Klein and Valentiner respectively. He also attempted to determine all the primitive substitution groups which can be represented on no more than 17 letters, but in this enumeration he also omitted a number of important groups. In particular he found only 14 out of 22 primitive groups of degree 16.

Errors in Definitions.—The fact that early writers generally did not use a precise definition of group is a source of considerable confusion.¹ Even in such a good work as Klein's *Ikosæder*, page 5, we read that a set of operations constitute a group when the composition of two of them gives one of the set. It is, of course, true that such a general definition may be used, and one finds authority for it in the French encyclopedia, tome 1, volume 2, page 243. In fact, the term group is defined in two very distinct ways in this standard work of reference. A commonly accepted definition is found on page 576 of the first volume. It seems to the writer unfortunate that two distinct definitions of this widely used term should appear in a work of reference, since this may tend to perpetuate its use in a vague manner by careless writers.

A FEW RECENT ERRORS IN GROUP THEORY.

Errors in the Encyclopædias.—A queer historical error occurs under the word "group" in the latest edition of the *Encyclopædia Britannica*. It is stated here that the mathematical technical sense of the word group is not older than 1870. If we recall that Cayley used the term group with its technical meaning in the heading of a number of his articles, beginning as early as 1854; and that Kirkman and Sylvester also used it about 1860, it appears very strange that an Englishman should have made the above statement. Numerous other errors relating to the history of mathematics occur in the latest edition of this popular work of reference but none of them appears less excusable than the one which has just been noted.

That the great French mathematical encyclopedia is not free from errors in its article on group theory may readily be seen from the many corrections which have appeared in the "Tribune publique." The German mathematical encyclopedia devoted such a small amount of space to the subject of finite groups that one could scarcely expect many errors. There are, however, also several somewhat serious errors in this work. Perhaps the most important one occurs on page 221, where it is stated, in substance, that the holomorph of a group is its group of isomorphisms, although the term holomorph is not used.

A very important recent general article on groups of finite order appeared in the latest edition of *Pascal's Repertorium*. This article is also unusually

¹ The present abstract definition is due to Germans; especially to Kronecker (1870), Weber (1882), and Frobenius (1887).

free from errors, notwithstanding its broad scope. One of the most important errors in it occurs on page 229, where the fundamental theorem, that with each regular group there may be associated a similar regular group in the same letters such that each of these groups is composed of all the substitutions in these letters which are commutative with every substitution of the other, is credited to Cayley (1878). As a matter of fact Jordan proved this theorem in his thesis published in 1860 and republished in *L'École Polytechnique* during the following year.

Error of Weber.—A very interesting error occurs in volume 2, page 54, of the first edition of Weber's *Lehrbuch der Algebra*. It is stated here that the most important example of a commutative group is furnished by the natural numbers when they are combined by ordinary multiplication. As a matter of fact these numbers do not form a group at all, as Weber himself observes in the second edition of his classic algebra. This most important example of a commutative group is therefore no group. In this connection we may refer to a historic misstatement made by Weber several years ago and repeated by many others. In his biography of Kronecker, which appeared at the beginning of volume 43 of the *Mathematische Annalen*, Weber states that Kronecker mentions Galois for the first time in a letter to Dirichlet in 1856. On the other hand, Kronecker had referred several times to Galois' theory in an article which was read several years earlier and appeared in the *Berichte* of the Berlin Academy for 1853.

Inconsistent Definitions.—An instance where the technical and the non-technical meaning of the word group are confused may deserve notice here, since it occurs in a very useful reference work; namely, in volume I of the *Subject Index* of the Royal Society of London Catalogue of Scientific Papers, which was published in 1908. On page 66 of this index we find a reference to Cockle's papers on "Vanishing Groups." This reference appears under the technical subject groups, but the papers by Cockle refer entirely to groups in the non-technical sense and have therefore nothing to do with groups as a technical concept.

The fact that the same name is assigned to two distinct groups in two well-known works of reference is a source of some confusion. In volume I, page 238, of the latest edition (1910) of *Pascal's Repertorium*, the symmetric group of order 6 is called the *anharmonic group*, while Capelli uses the same name for an entirely different group in the most recent edition of his algebra, entitled, *Istituzioni di analisi algebrica*, 1909, page 111. It may also be observed that some writers use the term degree for what is commonly known as the order of a group, and that the term metacyclic is used with different meanings.

Lack of References.—The literature of group theory has been made unnecessarily complicated by a lack of a proper appreciation of the importance of due references. In fact, some writers have not only failed to give proper references but have not even taken the trouble to find out whether their papers contained any new results before offering them for publication. In many cases such errors are not noted even in the reviews which appear in such good works as the *Jahrbuch ueber die Fortschritte der Mathematik*.

As an instance of this kind we may cite an article by Netto which appeared in *Crelle's Journal*, volume 128 (1905), page 243. We find in this article a development of the tetrahedral group according to an abstract definition which has been well known for more than half a century, and yet no references are given by the author of the article in question or by the reviewer in the *Fortschritte*. The article contains a number of other things which had been published several years before this article appeared. An instance of an article which contained nothing new and yet appeared in the excellent *Mathematische Annalen*, without proper reference, may be found by consulting this journal, volume 60 (1905), page 319.

Errors of Judgment.—There is a class of indirect errors to which it may be desirable to refer here; namely, those which arise from undue emphasis on some particular phase of the subject or on the work of some particular man. According to the writer's opinion Burnside's *Theory of Groups* errs along this line by quoting Hölder's work relatively too frequently. In fact, these references are not always correct. Even in the second edition, page 39, we are told that the symbol for quotient group was introduced by Hölder, although this symbol had been employed by Jordan at a much earlier date. As other instances of undue emphasis in important publications with respect to the work of one man, we may cite the references to Frobenius in the excellent article on groups contained in *Pascal's Repertorium* and the references to the present writer in the French encyclopedia article on this subject.

Conclusion.—It may be added that the above are only a few typical examples of the many errors which have been committed by some of those who have contributed much to the development of group theory. A complete collection of such errors, with proper explanations, would make a volume. Some of these errors have not had an unmitigatedly baneful effect, as they inspired young investigators with confidence, when these found that they could correct mistakes in the works of eminent mathematicians. The contributions to mathematics, due to the inspiration received from the discovery of errors in the works of eminent men, have been of great value to our science; but the beauty and insight furnished by correct and fundamental results are still more inspiring and are much more worthy objects of our endeavors.

ON THE REMAINDER TERM IN A CERTAIN DEVELOPMENT OF $f(a + x)$.*

By R. D. CARMICHAEL, Indiana University.

In the *Annals of Mathematics*, Vol. 5, No. 4, July, 1904, Mr. S. A. Corey gave an interesting and important development of $f(a + x)$, and showed that his formula was admirably adapted to the numerical computation of complicated

* Read before the American Mathematical Society, February 29, 1908.